

Spherically symmetric massive scalar fields in GR

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First we review some of the attempts made to find exact spherically symmetric solutions of Einstein field equations in the presence of scalar fields. Wyman solution in both static and non static scalar field is discussed briefly and it is shown that why in the case of non static homogeneous matter field, static metric can not be represented in terms of elementary functions. We mention here that if our spacetime be static, according to EFE there is two option for choose scalar field matter: static (time independent) and non static (time dependent). All these solutions are limited to the minimally coupled massless scalar fields and also in the absence of the cosmological constant. Then we show that if we are interesting to have a homogeneous isotropic scalar field matter one can construct a series solution in terms of scalar fields mass and cosmological constant. This metric is static and posses a locally flat case as a special chooses of mass of scalar field and can be interpreted as an effective vacuum. Therefore mass of scalar field eliminates any locally gravitational effect as tidal forces. Finally we describe why this system is unstable in the language of dynamical systems.

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I. BACKGROUND

Static spherically symmetric spacetimes are the simplest ones to study. Bergmann and Leipnik [1] are the first who studied solvability of field equations in the presence of scalar fields. They were not able to find an explicit solution due to their choice of a Schwarzschild like coordinate. Buchdahl in a series of papers [24] introduced and developed the reciprocal metrics idea as a generating method to construct new vacuum or non vacuum static solutions as a simple transformation of a known vacuum or non vacuum one. The higher dimensional extension of Buchdahl family has been introduced by Xanthopoulos and Zannias [5]. Later Wyman [6] attacked this problem in a general framework and showed that apart from the Buchdahl family of solutions there is another family of static spherically symmetric solutions in which the scalar field is not static. Later critical phenomena in gravitational collapse were discovered by Choptuik [1315] in the model of a spherically symmetric, massless scalar field minimally coupled to general relativity. The scalar field matter is both simple, and acts as a toy model in spherical symmetry for the effects of gravitational radiation. Important analytical studies of gravitational collapse in this model have been carried out by Christodoulou [1622]. For other numerical work on this model, see [2327]. In this work after a brief review of some of the previous studies we show that there is no closed form of Wyman class and then in section III first we derive EFE for a massive minimally coupled scalar field in the presence of cosmological constant in a spherically symmetric space time. Then by supposing that our scalar field has small time variations such that we can disregard its time dependency, and corresponding to it a static metric, we derive a series solution for our model. We mention here that if we assume that our field remains static, pressure and energy density describe a stiff matter and our spacetime posses homogeneity. This series solution has positive spatial parity and by choosing a suitable mass of scalar field is locally flat. Finally we study the stability of these solutions as a dynamical system and as an important result we show that Wyman minimally coupled solutions is unstable in the language of dynamical systems.

II. NON INTEGRABILITY OF NON STATIC SCALAR FIELDS IN SPHERICALLY SYMMETRIC SPACETIMES

We start from the Lagrangian of a static massless minimally coupled spherically symmetric scalar field with action:

$$S = \int (-g)^{\frac{1}{2}} (R + \mu g^{\mu\nu} \phi_{;\mu} \phi_{;\nu}) d^4x \quad (1)$$

where in it ($\mu, \nu = 0, 1, 2, 3$). We can write the static spherically symmetric solution in schwarzschild gauge as:

$$ds^2 = -e^\nu dt^2 + e^\lambda dr^2 + r^2(d\vartheta^2 + \sin^2\vartheta d\varphi^2) \quad (2)$$

Einstein and scalar field equations are:

$$\square\phi = 0 \quad (3)$$

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = -\mu(\phi_{;\mu}\phi_{;\nu} - \frac{1}{2}g_{\mu\nu}\phi_{;\alpha}\phi^{;\alpha}) \quad (4)$$

Buchdahl in series of papers [3, 4] introduced a new transformation which we called reciprocal transformation. Imposing Buchdahl transformations to the Schwarzschild as the seed metric, we end up with a one parameter family of exact solution for a minimally coupled massless static scalar field. As another example of the same technique a static solution with axial symmetry in weyl coordinates can be obtained [5] starting from the Weyl solution. The one parameter Buchdahl solution in spherically symmetric coordinate is given by:

$$\begin{aligned} ds^2 &= -(1 - \frac{2M}{r})^\beta dt^2 + (1 - \frac{2M}{r})^{-\beta} dr^2 + r^2(d\vartheta^2 + \sin^2\vartheta d\varphi^2) \\ \phi &= \lambda \ln(1 - \frac{2M}{r}) \\ \beta &= \pm \sqrt{1 - 2\mu\lambda^2} \end{aligned} \quad (5)$$

Field equations are now simplified to;

$$R_{\mu\nu} = -\mu\phi_{;\mu}\phi_{;\nu} \quad (6)$$

Wyman showed that [6] in Buchdahl formalism one family of solutions is missing and Yilmaz [7], Szekeres [8] and Buchdahl solutions are all sub classes of his static family. Here we restrict ourselves to Wyman's non static spherically symmetric scalar field family of solution ; i.e the case in which $\lambda = \lambda(r), \nu = \nu(r)$, $\phi = \phi(t)$ and the metric functions (2) satisfy the following differential equation equations:

$$\begin{aligned} \dot{\nu} + \dot{\lambda} &= \mu r e^{\lambda-\nu} \\ \dot{\nu} - \dot{\lambda} &= 2(1 - e^\lambda)/r \end{aligned} \quad (7)$$

Eliminating ν we find the following for λ ;

$$\lambda'' + \frac{3\lambda'}{r}(e^\lambda - 1) + \frac{2}{r^2}(e^\lambda - 1)(e^\lambda - 2) = 0 \quad (8)$$

Wyman has stated that the above equation (8) can not be integrated and by imposing suitable boundary conditions derived a series solution for the metric functions. Now using results from differential equations we show that this family could not be written in closed form. Using the following change of variables :

$$r = e^t, u = \dot{\lambda} = \frac{d\lambda}{dt}, x = e^\lambda, \quad (9)$$

Equation (8) is transformed into:

$$u \frac{du}{dx} = u \frac{4-3x}{x} - \frac{2}{x}(x-1)(x-2) \quad (10)$$

This differential equation is of 2-nd Abel type (class A) [9]. It is well known that for a general equation of the form:

$$(y(x) + g(x)) \frac{dy}{dx} = f_2(x)y(x)^2 + f_1(x)y(x) + f_0(x) \quad (11)$$

There are solutions only in the following two special cases:

Case(1): If

$$\begin{aligned} f_0(x) &= f_1(x)g(x) - f_2(x)g(x)^2 \\ y(x) &= -g(x) \\ y(x) &= e^{\int f_2(x)dx} (c_1 + \int (-f_2(x)g(x) + f_1(x))e^{-\int f_2(x)dx} dx) \end{aligned} \quad (12)$$

Case(2): If

$$\begin{aligned}
 f_1(x) &= 2f_2(x)g(x) - g(x)g'(x) \\
 \alpha(x) &= e^{-2 \int f_2(x) dx} \\
 \beta(x)^2 &= \alpha(x)^2 g(x)^2 + 2\alpha(x)c_1 + 2\alpha(x) \int \frac{f_0(x)dx}{(\int dx e^{\int f_2(x) dx})^2} \\
 y(x) &= -\frac{\alpha(x)g(x) + \sqrt{\beta(x)}}{\alpha(x)}
 \end{aligned} \tag{13}$$

Since Wyman equation does not belong to any of these cases, one can not derive an exact closed form for it.

In this section we obtain a new series solution for a massive-minimally coupled scalar field in the presence of a cosmological constant. According to Wyman we take our spherically symmetric static spacetime metric in Schwarzschild gauge, and require the metric functions to be regular near the origin. So that the scalar field and metric functions have a Taylor expansion near the origin. The field equation are now given by:

$$\begin{aligned}
 R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} - \Lambda g_{\mu\nu} &= -\kappa T_{\mu\nu} \\
 \kappa T_{\mu\nu} &= -\mu[\phi_{;\mu}\phi_{;\nu} - \frac{1}{2}g_{\mu\nu}(\phi_{;\alpha}\phi^{;\alpha} - m^2\phi^2)]
 \end{aligned} \tag{14}$$

Where we have introduced a parameter μ in the energy-momentum tensor to make sure one that our solution reduces to the vacuum solution in the absence of the scalar field. we take it 1 for simplicity. The equivalent equation can be written as:

$$R_{\mu\nu} = -\Lambda g_{\mu\nu} - \phi_{;\mu}\phi_{;\nu} + \frac{1}{2}m^2\phi^2 g_{\mu\nu} \tag{15}$$

Assuming that our spherically symmetric metric in the the Schwarzschild gauge is of the general form:

$$ds^2 = e^\nu dt^2 - e^\lambda dr^2 - r^2(d\theta^2 + \sin^2\theta d\phi^2) \tag{16}$$

The scalar field equation of motion is:

$$\square\phi + m^2\phi = 0, \tag{17}$$

and equation (15) reduces to the following differential equations:

$$\begin{aligned}
 -\frac{1}{4r}e^{-\lambda}(2r\nu'' + r(\nu')^2 - r\nu'\lambda' + 4\nu') &= -\Lambda + \frac{1}{2}m^2\phi^2 \\
 \frac{1}{4r}(2r\nu'' + r(\nu')^2 - r\nu'\lambda' - 4\lambda') &= \Lambda e^\lambda - \dot{\phi}^2 - \frac{1}{2}m^2\phi^2 e^\lambda \\
 \frac{1}{2}e^{-\lambda}(-r\lambda' - 2e^\lambda + r\nu' + 2) &= -\Lambda r^2 - \frac{1}{2}m^2\phi^2 r^2 \\
 \frac{d^2\phi}{dr^2} + 2\left(\frac{1}{r} + \frac{\nu' - \lambda'}{2}\right)\frac{d\phi}{dr} - m^2e^\lambda\phi &= 0
 \end{aligned} \tag{18}$$

From (18) after some little algebra we find:

$$\begin{aligned}
 \dot{\nu} + \dot{\lambda} &= -r\dot{\phi}^2 \\
 \dot{\nu} - \dot{\lambda} &= \frac{2}{r}[e^\lambda(1 - \Lambda r^2) - 1 - \frac{1}{2}m^2\phi^2 r^2 e^\lambda]
 \end{aligned} \tag{19}$$

There is no simple reduction of this system to a one variable solvable differential equation. Actually field equation can be written in the following complicated non linear form:

$$\dot{\phi}^2 = \frac{h_3\phi'''' + 3h_2\phi''}{h_0}, \tag{20}$$

in which:

$$\begin{aligned} h_0 &= r\phi''(rm^2\phi + \phi'(-2 + 2\Lambda r^2 + m^2r^2\phi^2)) \\ h_3 &= (-2 + 2\Lambda r^2 + m^2r^2\phi^2)\phi'' + \frac{4}{3}m^2r^2\phi\phi' + \frac{4}{3}r\phi'(m^2\phi^2 + \frac{m^2}{2} + 2\Lambda) + \frac{2}{3}m^2\phi \\ h_2 &= (4 - 4\Lambda r^2 - 2m^2r^2\phi^2)\phi' - 2m^2r\phi \end{aligned} \quad (21)$$

It does not seem one could be able solve (20) analytically, so we try a semi-analytic solution by imposing the same simple boundary conditions on metric as Wyman did [6]:

$$\nu(0) = 0, \lambda(0) = 0, \phi(0) = q, \phi'(0) = 0 \quad (22)$$

These boundary conditions are chosen such that we have a regular solution at the origin. q is interpreted as a new parameter of our two parameter (q, m) family of exact solutions, apart from the mass of the scalar field and the cosmological constant. Applying the above boundary conditions in equation (19) and taking successive derivatives of equations and evaluating them at the origin we can write the following series for the scalar field and the metric functions [12]:

$$\begin{aligned} \phi(r) &= q(1 + \sum_{n=2,4,6,\dots} a_n(q^2)r^n) \\ e^\nu &= 1 - \sum_{n=2,4,6,\dots} b_n(q^2)r^n \\ e^\lambda &= 1 + \sum_{n=2,4,6,\dots} c_n(q^2)r^n \end{aligned} \quad (23)$$

We note that only even values of n contribute in the solution. This means that our functions remain unchanged under parity transformation $r \rightarrow -r$. We can conclude that our metric functions can be written in general form $X(r) = 1 \pm f(m^2q^2, \Lambda r^2)$. The first few values of the coefficients in (23) are give below:

$$\begin{aligned} a_2(x) &= \frac{1}{6}m^2 \\ a_4(x) &= \frac{1}{4!}(\frac{1}{3}m^4x + \frac{1}{5}m^4 + \frac{2}{3}\Lambda m^2) \\ b_2(x) &= \frac{1}{6}[m^2x + 2\Lambda] \\ b_4(x) &= \frac{1}{60}[m^4x] \\ c_2(x) &= \frac{1}{6}[m^2x + 2\Lambda] \\ c_4(x) &= \frac{1}{24}[\frac{16}{15}m^2x + \frac{2}{3}m^4x^2 + \frac{8}{3}m^2\Lambda x + \frac{8}{3}\Lambda^2] \end{aligned} \quad (24)$$

If $q = 0$ the solution reduces to that of the series expansion for de-Sitter solution. It is clear that there is no consistent Wyman solution of 2-nd class. Calculating all the Rieman Tensor components we observe that the only non zero component up to 4-th order in r (near the origin) is given by:

$$R_{101}^0 = \frac{1}{6}\mu m^2 q^2 - \frac{1}{3}\Lambda \quad (25)$$

So if the mass of the scalar field satisfies $m = q^{-1}\sqrt{2\Lambda}$ this component vanishes and the spacetime becomes flat and every local effect of gravity as tidal forces vanishes.

MASSIVE SCALAR FIELD AS A DYNAMICAL SYSTEM

In this section we apply a powerful method to analyze the stability of the scalar field solutions obtained in the previous section. This method for an autonomous ODE has already been employed by Coley [10] in the context of

cosmology. To do so we introduce some concepts from nonlinear complex systems [11]. In the language of dynamical systems our differential equation could be represented as follows :

$$\begin{aligned}\dot{x} &= f(t, x) \\ f &: [0, \infty) \times D \longrightarrow \mathbb{R}^n \\ D &= \{x \in \mathbb{R}^n \mid \|x\|_2 < 0\}\end{aligned}\tag{26}$$

We also assume that $x = 0$ be the equilibrium point of the system at $t = 0$ that is:

$$f(t, 0) = 0, \forall t \geq 0\tag{27}$$

We assume that the Jacobian matrix $[\frac{\partial f}{\partial x}]$ is bounded with respect to t on the set D and smoothly satisfies Lipschitz lemma, that is:

$$\|f(t, x) - f(t, y)\| \leq L \|x - y\|, \|x\|_p = \left(\sum_i^p |x_i|^p\right)^{\frac{1}{p}}, 1 \leq p < \infty\tag{28}$$

One can state the following theorems about this system.

Theorem (1): In the neighborhood of any stability point the system can be approximated by a linear one i.e.:

$$\dot{x} = A(t)x\tag{29}$$

Theorem (2): Assume that $x = 0$ be an equilibrium point of the nonlinear system $\dot{x} = f(t, x)$ that is :

$$\begin{aligned}f &: [0, \infty) \times D \longrightarrow \mathbb{R}^n \\ D &= \{x \in \mathbb{R}^n \mid \|x\|_2 < r\},\end{aligned}\tag{30}$$

and define the matrix; $A(t) = \frac{\partial f(t, x)}{\partial x}|_{x=0}$. If the origin is an exponential stable equilibrium point of the linear system (29) then this point is the exponential stable equilibrium point of the non-linear system (26).

Now we apply these theorems to our system. First we rewrite field equations through the following variables.;

$$x_1 = \nu(r), x_2 = \lambda(r), x_3 = \phi(r), x_4 = \frac{d\phi(r)}{dr},\tag{31}$$

as follows

$$\begin{aligned}x_1' &= \frac{e^{x_2} - 1}{r} + \Lambda r e^{x_2} - \frac{1}{2}r(x_4^2 + m^2 e^{x_2} x_3^2) \\ x_2' &= -\frac{e^{x_2} - 1}{r} - \Lambda r e^{x_2} - \frac{1}{2}r(x_4^2 - m^2 e^{x_2} x_3^2) \\ x_3' &= x_4 \\ x_4' &= \left[\frac{-e^{x_2} + 1}{r} - \Lambda r e^{x_2} + \frac{1}{2}m^2 r x_3^2 e^{x_2}\right]x_4 - m^2 x_3 e^{x_2}\end{aligned}\tag{32}$$

Equilibrium point of the above system lies at:

$$x_1 = x_1, x_2 = -\ln(1 + \Lambda r^2), x_3 = 0, x_4 = 0\tag{33}$$

This point of 4-dimensional phase space corresponds to a $t = cte$ hypersurface with constant negative intrinsic curvature $k = -\Lambda$ with metric:

$$ds_{t=cte}^2 = \frac{dr^2}{1 + \Lambda r^2} + r^2(d\theta^2 + \sin^2\theta d\varphi^2)\tag{34}$$

We assume that the equilibrium point (solution) is perturbed as $x_i = x_i^0 + u_i$. The Jacobian matrix can be easily calculated and the result is:

$$A_{ij} = \left[\frac{\partial f_i}{\partial x_j}(t, x)\right] = \begin{bmatrix} 0 & r^{-1} & 0 & 0 \\ 0 & r^{-1} & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & \frac{m^2}{1 + \Lambda r^2} & 0 \end{bmatrix}\tag{35}$$

We can find an exact solution for perturbations of our dynamical system variables (29) according to the following system of ordinary differential equations, using (29) ;

$$\begin{aligned}
u_1' &= \frac{1}{r} u_2 \\
u_2' &= \frac{1}{r} u_2 \\
u_3' &= u_4 \\
u_4' &= \frac{m^2}{1 + \Lambda r^2} u_3
\end{aligned} \tag{36}$$

From these set of equations we immediately obtain:

$$\begin{aligned}
u_1 &= c_1 r + c_2 \\
u_2 &= c_1 r \\
u_3 &= (1 + \Lambda r^2)(c_1 F([a_1, a_2], 1/2, -\Lambda r^2) + \\
&\quad c_2 F([a_1 + 1/2, a_2 + 1/2], 1/2, -\Lambda r^2)) \\
u_4 &= \frac{du_4}{dr}
\end{aligned} \tag{37}$$

In which:

$$\begin{aligned}
F(n, d, z) &= \sum_{k=0}^{\infty} \frac{z^k}{k!} \prod_{i=1}^p (n_i)_k \prod_{j=1}^q (d_j)_k \\
(d_j)_k &= \frac{\Gamma(k + d_j)}{\Gamma(d_j)} \\
n &= [n_1, n_2, \dots], p = nops(n), d = [d_1, d_2, \dots], q = nops(d) \\
a_1 &= 1/4 \frac{3\sqrt{\Lambda} + i\sqrt{4m^2 - \Lambda}}{\sqrt{\Lambda}} \\
a_2 &= -1/4 \frac{-3\sqrt{\Lambda} + i\sqrt{4m^2 - \Lambda}}{\sqrt{\Lambda}},
\end{aligned} \tag{38}$$

and the symbol *nops* means the number of operands of an expression. If some n_i is a non-positive integer, the series is finite (that is, $F(n, d, z)$ is a polynomial in z). If some d_j is a non-positive integer, the function is undefined for all non-zero z , unless there is also a negative upper parameter of smaller absolute value, in which case the previous rule applies. For the remainder of this description, assume no n_i or d_j is a non-positive integer. When $p \leq q$, this series converges for all complex z , and hence defines $F(n, d, z)$ everywhere. When $p = q + 1$, the series converges for $|z| < 1$. $F(n, d, z)$ is then defined for $|z| \geq 1$ by analytic continuation. The point $z = 1$ is a branch point, and the interval $(1, \infty)$ is the branch cut. When $p > q + 1$ the series diverges for all $z \neq 0$. In this case, the series is interpreted as the asymptotic expansion of $F(n, d, z)$ around $z = 0$. The positive real axis is the branch cut.

Now the main result comes from the fact that the solution (37) is not asymptotically convergent i.e;

$$u_i(r \rightarrow \infty) = \infty \tag{39}$$

So although we treated the system in the linear approximation, we deduce that it is not stable. Therefore it is guaranteed that our nonlinear system is not stable too. For a massless (Wyman) solution in the absence of the cosmological constant we can refer to the system of equations (36) and its solutions (37). Obviously in this case the perturbations diverge and so this family is not stable as well.

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S. Wiggins, Introduction to Applied Non linear Dynamical Systems and chaos (Springer 1990).
- [12] We take limit from both sides of equations(19) and using boundary conditions mentioned in (22) we deduced that $\dot{\nu}(0) = \dot{\lambda}(0) = 0$. Also from equation (18) we immediately obtain $\phi''(0) = \frac{1}{3}m^2q$. Differentiation from equations(19) with respect to r and taking limit $r \rightarrow 0$ we obtain $\nu''(0) = \frac{-1}{3}\mu m^2q^2 + \frac{2}{3}\Lambda$, $\lambda''(0) = \frac{1}{3}\mu m^2q^2 - \frac{2}{3}\Lambda$, $\phi'''(0) = 0$. We seek that for all functions of our model $X^{2n+1}(0) = 0, n \geq 0$. Unfortunately since there is no simple one variable differential equation which from it, we can obtain a recursion relation this assumption can not be proved analytically.